

DERIVING N -SOLITON SOLUTIONS VIA CONSTRAINED FLOWSYUNBO ZENG¹*Department of Mathematical Sciences,
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Abstract. The soliton equations can be factorized by two commuting x - and t -constrained flows. We propose a method to derive N -soliton solutions of soliton equations directly from the x - and t -constrained flows.

Keywords: constrained flow, soliton equation, soliton solution

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1. Introduction.

In recent years much work has been devoted to the constrained flows of soliton equations (see, for example, [1-7]). It was shown in [1-3] that (1+1)-dimensional soliton equation can be factorized by x - and t -constrained flow which can be transformed into two commuting x - and t -finite-dimensional integrable Hamiltonian systems. The Lax representation for constrained flows can be deduced from the adjoint representation of the auxiliary linear problem for soliton equations [4]. By means of the Lax representation, the standard method in [8-10] enables us to introduce the separation variables for constrained flows [11-15] and to establish the Jacobi inversion problem [13-15]. Finally, the factorization of soliton equations and separability of the constrained flows allow us to find the

Jacobi inversion problem for soliton equations [13-15]. By using the Jacobi inversion technique [16,17], the N -gap solutions in term of Riemann theta functions for soliton equations can be obtained, namely, the constrained

flows can be used to derive the N -gap solution for soliton equations. It has been believed that the constrained flows can also been used directly to derive the N -soliton solutions for soliton equations. However this case remains a challenging problem.

It is well known that there are several methods to derive the N -soliton solution of soliton equations, such as the inverse scattering method, the Hirota method, the dressing method, the Darboux transformation, etc.

(see, for example, [18-20] and references therein). In present paper, we propose a method to construct directly N -soliton solution from two commuting x - and t -constrained flows. We will illustrate the method by KdV equation. The method can be applied to other soliton equations.

2. Constrained flows.

We first recall the constrained flows and factorization of soliton equations by using KdV equation. Let consider the Schrödinger spectral problem

$$-\phi_{xx} + u\phi = \lambda\phi. \quad (2.1)$$

The KdV hierarchy associated with (2.1) can be written in infinite-dimensional integrable Hamiltonian system [18-20]

$$u_{t_n} = \partial_x \frac{\delta H_n}{\delta u}, \quad n = 1, 2, \dots, \quad (2.2)$$

where

$$\frac{\delta H_n}{\delta u} = L^n u, \quad L = -\partial_x^2 + 4u - 2\partial_x^{-1}u_x, \quad \partial_x^{-1}\partial_x = \partial_x\partial_x^{-1} = 1. \quad (2.3)$$

The well known KdV equation reads

$$u_t - 6uu_x + u_{xxx} = 0. \quad (2.4)$$

For KdV equation (2.4), the time evolution equation of ϕ is given by

$$\phi_t = 4\lambda\phi_x + 2u\phi_x - u_x\phi. \quad (2.5)$$

The compatibility condition of (2.1) and (2.5) gives rise to (2.4).

It is known that

$$\frac{\delta\lambda}{\delta u} = \phi^2. \quad (2.6)$$

The constrained flows of the KdV hierarchy consists of the equations obtained from the spectral problem (2.1) for N distinct real numbers λ_j and the restriction of the variational

derivatives for the conserved quantities H_{k_0} (for any fixed k_0) and λ_j [2-4]

$$-\phi_{j,xx} + u\phi_j = \lambda_j\phi_j, \quad j = 1, \dots, N, \quad (2.7a)$$

$$\frac{\delta H_{k_0}}{\delta u} - \sum_{j=1}^N \alpha_j \frac{\delta \lambda_j}{\delta u} = 0. \quad (2.7b)$$

The system (2.7) is invariant under all the KdV flows (2.2).

For $k_0 = 0$, in order to obtain N -soliton solution, we take

$$\lambda_j < 0, \quad \zeta_j = \sqrt{-\lambda_j}, \quad \alpha_j = 4\zeta_j, \quad j = 1, \dots, N,$$

one gets from (2.7b)

$$u = 4 \sum_{j=1}^N \zeta_j \phi_j^2 = 4\Phi^T \Theta \Phi, \quad (2.8)$$

where

$$\Phi = (\phi_1, \dots, \phi_N)^T, \quad \Theta = \text{diag}(\zeta_1, \dots, \zeta_N), \quad \Lambda = \text{diag}(\lambda_1, \dots, \lambda_N).$$

By substituting (2.8), (2.7a) becomes

$$-\phi_{j,xx} + 4 \sum_{i=1}^N \zeta_i \phi_i^2 \phi_j = \lambda_j \phi_j, \quad j = 1, \dots, N,$$

or equivalently

$$\Phi_{xx} = -\Lambda\Phi + 4\Phi\Phi^T\Theta\Phi. \quad (2.9)$$

After inserting (2.8), (2.5) reads

$$\Phi_t = 4\Lambda\Phi_x + 8\Phi_x\Phi^T\Theta\Phi - 8\Phi\Phi^T\Theta\Phi_x. \quad (2.10)$$

The compatibility of (2.7), (2.10) and (2.4) ensures that if Φ satisfies two compatible systems (2.9) and (2.10), simultaneously, then u given by (2.8) is a solution of KdV equation (2.4), namely, the KdV equation (2.4) is factorized by the x -constrained flow (2.9) and t -constrained flow (2.10).

The Lax representation for the constrained flows (2.9) and (2.10), which can be deduced from the adjoint representation of the spectral problem (2.1) by using the method in [3,4], is given by

$$Q_x = [\tilde{U}, Q],$$

where \tilde{U} and the Lax matrix Q are of the form

$$\tilde{U} = \begin{pmatrix} 0 & 1 \\ -\lambda + 4\Phi^T \Theta \Phi & 0 \end{pmatrix}, \quad M = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix},$$

$$A(\lambda) = -2 \sum_{j=1}^N \frac{\zeta_j \phi_j \phi_{j,x}}{\lambda - \lambda_j}, \quad B(\lambda) = 1 + 2 \sum_{j=1}^N \frac{\zeta_j \phi_j^2}{\lambda - \lambda_j},$$

$$C(\lambda) = -\lambda + 2\Phi^T \Theta \Phi - 2 \sum_{j=1}^N \frac{\zeta_j \phi_{j,x}^2}{\lambda - \lambda_j}.$$

Then $\frac{1}{2} \text{Tr} M^2(\lambda) = A^2(\lambda) + B(\lambda)C(\lambda)$, which is a generating function of integrals of motion for the system (2.9) and

(2.10), gives rise to

$$A^2(\lambda) + B(\lambda)C(\lambda) = -\lambda - 2 \sum_{j=1}^N \frac{F_j}{\lambda - \lambda_j},$$

where $F_j, j = 1, \dots, N$, are N independent integrals of motion for the systems (2.9) and (2.10)

$$F_j = \phi_{j,x}^2 + (\lambda_j - 2 \sum_{i=1}^N \zeta_i \phi_i^2) \phi_j^2 + 2 \sum_{k \neq j} \frac{\zeta_k (\phi_{j,x} \phi_k - \phi_j \phi_{k,x})^2}{\lambda_j - \lambda_k}, \quad j = 1, \dots, N. \quad (2.11)$$

3. Deriving N -soliton solution.

In order to constructing N -soliton solution, we have to set $F_j = 0$. It follows from (2.9) that

$$\frac{\phi_{j,x} \phi_k - \phi_j \phi_{k,x}}{\lambda_j - \lambda_k} = -\partial_x^{-1}(\phi_j \phi_k). \quad (3.1)$$

Then one gets

$$F_j = \phi_{j,x}^2 + (\lambda_j - 2 \sum_{i=1}^N \zeta_i \phi_i^2) \phi_j^2 - 2 \sum_{k=1}^N \zeta_k (\phi_{j,x} \phi_k - \phi_j \phi_{k,x}) \partial_x^{-1} (\phi_j \phi_k) = 0, \quad j = 1, \dots, N. \quad (3.2)$$

The integrals of motion F_j can be used to reduce the order of system (2.9). By multiplying (2.9) by ϕ_j and adding it to (3.2), one obtains

$$-\phi_j [\phi_{j,x} - 2 \sum_{k=1}^N \zeta_k \phi_k \partial_x^{-1} (\phi_j \phi_k)]_x + \phi_{j,x} [\phi_{j,x} - 2 \sum_{k=1}^N \zeta_k \phi_k \partial_x^{-1} (\phi_j \phi_k)] = 0, \quad j = 1, \dots, N,$$

which results to

$$\phi_{j,x} - 2 \sum_{k=1}^N \zeta_k \phi_k \partial_x^{-1} (\phi_j \phi_k) = -\gamma_j \phi_j, \quad \gamma_j = \gamma_j(t), \quad j = 1, \dots, N,$$

or equivalently

$$\Phi_x = -\Gamma \Phi + 2\partial_x^{-1} (\Phi \Phi^T) \Theta \Phi, \quad (3.3)$$

where $\Gamma = \text{diag}(\gamma_1, \dots, \gamma_N)$. Set

$$R = 2\partial_x^{-1} (\Phi \Phi^T) \Theta, \quad (3.4)$$

Eq. (3.3) can be rewritten as

$$\Phi_x = -\Gamma \Phi + R \Phi. \quad (3.5)$$

Notice that

$$2\Phi \Phi^T = R_x \Theta^{-1}, \quad \Theta R = R^T \Theta, \quad (3.6)$$

it follows from (3.4) and (3.5) that

$$\begin{aligned} R_x &= 2\partial_x^{-1} (\Phi_x \Phi^T + \Phi \Phi_x^T) \Theta \\ &= 2\partial_x^{-1} (-\Gamma R_x + R R_x - R_x \Gamma + R_x R) = -\Gamma R - R \Gamma + R^2. \end{aligned} \quad (3.7)$$

We now show that

$$\gamma_j^2 = -\lambda_j, \quad \text{or} \quad \Gamma^2 = -\Lambda. \quad (3.8)$$

In fact, it is found from (3.5), (3.6) and (3.7) that

$$\begin{aligned} \Phi_{xx} &= -\Gamma \Phi_x + R \Phi_x + R_x \Phi = \Gamma^2 \Phi + (-\Gamma R - R \Gamma + R^2) \Phi + R_x \Phi \\ &= \Gamma^2 \Phi + 2R_x \Phi = \Gamma^2 \Phi + 4\Phi \Phi^T \Theta \Phi, \end{aligned}$$

which together with (2.9) leads to (3.8). Therefore, we can take $\Gamma = \Theta$, (3.5) and (3.7) can be rewritten as

$$\Phi_x = -\Theta\Phi + R\Phi, \quad (3.9)$$

and

$$R_x = -\Theta R - R\Theta + R^2, \quad (3.10)$$

$$2\Phi\Phi^T = R_x\Theta^{-1} = -\Theta R\Theta^{-1} - R + R^2\Theta^{-1}. \quad (3.11)$$

To solve (3.9), we first consider the linear system

$$\Psi_x = -\Theta\Psi. \quad (3.12)$$

It is easy to see that

$$\Psi = (c_1(t)e^{-\zeta_1 x}, \dots, c_N(t)e^{-\zeta_N x})^T. \quad (3.13)$$

Take the solution of (3.9) to be of the form

$$\Phi = \Psi - M\Psi,$$

then M has to satisfy

$$M_x = -\Theta M + M\Theta - R + RM. \quad (3.14)$$

Comparing (3.14) with (3.10), one finds

$$M = \frac{1}{2}R\Theta^{-1} = \partial_x^{-1}(\Phi\Phi^T). \quad (3.15)$$

So we have

$$\Phi = (I - M)\Psi = [I - \partial_x^{-1}(\Phi\Phi^T)]\Psi, \quad (3.16)$$

which leads to

$$\Psi = \sum_{n=0}^{\infty} M^n \Phi. \quad (3.17)$$

By using (3.15) and (3.17), it is found from that

$$\begin{aligned} \partial_x^{-1}(\Psi\Psi^T) &= \partial_x^{-1} \sum_{n=0}^{\infty} \sum_{l=0}^n M^l \Phi\Phi^T M^{n-l} \\ &= \partial_x^{-1} \sum_{n=0}^{\infty} \sum_{l=0}^n M^l M_x M^{n-l} = \sum_{n=1}^{\infty} M^n. \end{aligned} \quad (3.18)$$

Set

$$V = (v_{ij}) = \partial_x^{-1}(\Psi\Psi^T), \quad v_{ij} = -\frac{c_i(t)c_j(t)}{\zeta_i + \zeta_j} e^{-(\zeta_i + \zeta_j)x}. \quad (3.19)$$

One obtain

$$(I + V)\Phi = \Psi, \quad \text{or} \quad \Phi = (I - M)\Psi = (I + V)^{-1}\Psi. \quad (3.20)$$

By inserting (3.9) and (3.11), (2.10) becomes

$$\begin{aligned} \Phi_t &= [4\Theta^3 - 4\Theta^2 R + 8(-\Theta + R)\Phi\Phi^T\Theta - 8\Phi\Phi^T\Theta(-\Theta + R)]\Phi \\ &= 4\Theta^3\Phi - 4R\Theta^2\Phi. \end{aligned} \quad (3.21)$$

Let Ψ satisfy the linear system

$$\Psi_t = 4\Theta^3\Psi, \quad (3.22)$$

then

$$\Psi = (c_1(t)e^{-\zeta_1 x}, \dots, c_N(t)e^{-\zeta_N x})^T, \quad c_i(t) = \beta_j e^{4\zeta_j^3 t}, \quad j = 1, \dots, N. \quad (3.23)$$

We now show that Φ determined by (3.20) and (3.23) satisfy (3.21). In fact, we have

$$\begin{aligned} \Phi_t &= -(I + V)^{-1}V_t(I + V)^{-1}\Psi + (I + V)^{-1}\Psi_t \\ &= 4\Theta^3\Phi - 4M\Theta^3\Phi - 4(I - M)V\Theta^3\Phi \\ &= 4\Theta^3\Phi - 8M\Theta^3\Phi = 4\Theta^3\Phi - 4R\Theta^2\Phi. \end{aligned}$$

Therefore Φ given by (3.20) and (3.23) satisfies (2.9) and (2.10) simultaneously and $u = 4\Phi^T\Theta\Phi$ is the solution of KdV equation (2.4). It is easy

to show that this solution is just the N -soliton solution. Notice that

$$\begin{aligned} 2\partial_x(\Psi^T\Phi) &= -2\Psi^T\Theta\Phi + 2\Psi^T(-\Theta + R)\Phi \\ &= -4\Phi^T(I + V)(I - M)\Theta\Phi = -4\Phi^T\Theta\Phi, \end{aligned}$$

namely

$$u = -2\partial_x \sum_{i=1}^N c_i(t)e^{-\zeta_i x} \phi_i. \quad (3.24)$$

Formulas (3.20), (3.23) and (3.24) are just that obtained from the Gelfand-Levitan-Marchenko equation for determining the N -soliton solution for KdV equation [17,18,19] and finally results to the well-known expression for N -soliton solution of KdV equation (2.4)

$$u = -2\partial_x^2 \ln(\det(I + V)).$$

4. Conclusion.

The factorization of the KdV equation into two compatible x - and t -constrained flows enables us to derive directly the N -soliton solution via the x - and t -constrained flows. The method presented in this paper can be applied to other soliton equations for directly obtaining N -soliton solutions from constrained flows.

Acknowledgments.

This work was supported by the Chinese Basic Research Project "Nonlinear Sciences".

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